Computer Algebra Systems: Sophisticated 'Number Crunchers' or an Educational Tool for Learning to Think Mathematically?

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Abstract

The development of Computer Algebra Systems (CASs) in the last twenty years has yielded unprecedented progress in the ways in which mathematics can be used as a pedagogical tool. Unfortunately, for a variety of reasons, their acceptance as a viable learning aid has been cautiously received, particularly at tertiary level. Given the continuing concern regarding undergraduate instruction of mathematics, particularly at 'service subject' level, it is incumbent on instructors to explore the potential of CASs and investigate ways in which they can be integrated into their courses. This paper will present an outline of three approaches used by the author to include CASs in his classroom over a period of two years: using a CAS as a demonstration tool, using a CAS as an applications tool, and using a CAS as a self-paced exploratory tool. Underlying these methodological approaches are a number of issues that impact upon curriculum development and provide course planners with a means to review and refine already overcrowded syllabii. These include the ability of CASs to perform laborious computations, thus allowing the student to focus on conceptual understanding; the potential of CASs to extend students' understanding through transfer between algebraic and graphical representations of problems; and the role of visualization as a facilitator in the learning process. Central to the three approaches is the legitimacy of a curriculum which stresses the need for acquiring mathematical skills and techniques. If CASs are to be seriously accepted then new courses will need to be developed which stress underlying concepts. The implications of this for mathematics educators and software developers are discussed.

Keywords

Computer Algebra Systems, dual representations, visualization, exploration

1. Introduction

Computer Algebra Systems (CASs) are programming languages that can represent and manipulate numerical, symbolic and graphical information in a diverse range of topics within the mathematics curriculum. They were originally used as professional tools. However, since the mid 1980s, CASs have moved into both secondary and tertiary classrooms as viable learning aids. For a variety of reasons, their introduction, especially into undergraduate mathematics curricula has been unenthusiastic. As with the arrival of the calculator in the 1970s, there is a reticence to use technology. This could be due to a number of factors: it may be considered too cumbersome to implement or incorporate into existing syllabii; there may be uncertainty surrounding the educational outcomes; or even a perception that the technology may threaten the teacher's pedagogical importance. It is the purpose of this paper to dispel this myth, and to suggest that, when used thoughtfully, CASs can, in fact, *augment* learning and provide a rich and motivating environment to explore mathematics.

The most commonly used CASs in tertiary institutions are *MATHEMATICA*, *MAPLE*, and *DERIVE*. The author has used *DERIVE* (Rich *et al.*,1995) with first year undergraduate students for over two years. *DERIVE* is a simple, menu-driven application which can, within a brief period of instruction, provide students with a working knowledge of the software, thus allowing them to concentrate on the task of understanding their mathematics. Three *DERIVE* learning environments will be described in this paper:

- using *DERIVE* as a demonstration tool within a traditional lecture situation,
- using *DERIVE* as an 'applications' tool in a tutorial situation, and
- using *DERIVE* as a self-paced exploratory tool in a laboratory situation.

2. *DERIVE* as a Demonstration Tool

Through a laptop computer, projection pad, and an overhead projector, *DERIVE* was used to introduce, illustrate, and explore carefully selected concepts in precalculus and calculus to engineering mathematics students as part of their traditional lecture classes. Situations were presented which allowed the visual components of mathematics to be explained. For example, functions were represented symbolically and their various transformations analysed graphically through the '2D-Plot' facility. Figure 1 illustrates how the coefficients *'a'*, *'b'*, and *'c'* of the function $f(x) = ax^3 + bx^2 - cx + 1$ affect the graph's shape (Kutzler, 1994).

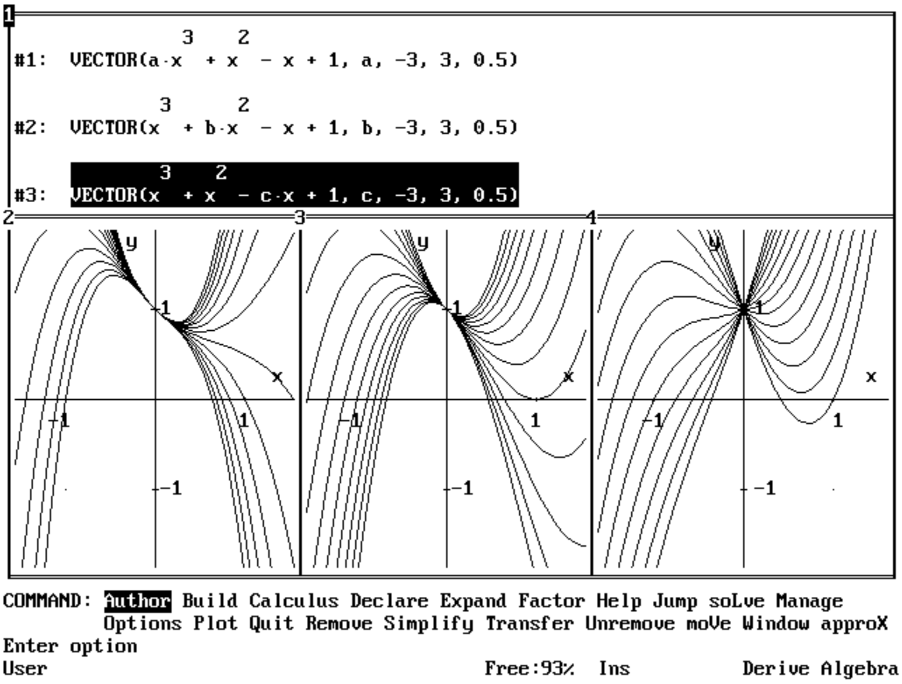

Figure 1. Analysing a polynomial through its symbolic and graphical representations

Numerous windows may be placed on the screen for students to view. The instructor can use the 'what if..?', 'why is that...?', and 'how can we summarise this so far...?' questioning routine to elicit information. Conversely, graphs can be generated on the screen and students prompted to speculate on their symbolic representation. Moving between symbolic and graphical representations is a relatively novel idea as an instructional exercise for students who are accustomed to the traditional 'sketch the following function...' protocol that dominates classroom practice at secondary level. The visual emphasis enforces the notion that functions have graphical representations and are not meaningless symbolic sentences.Students need to be encouraged to picture functions. This dual representation contributes to a greater awareness of the mathematics that they are performing and enables them to aquire an intuitive feel for the subject. This is further illustrated when introducing calculus concepts.

Sketching multiple tangents to the function $f(x) = x^2 - 1$ (figure 2), for instance, can provide a graphic (and aesthetically pleasing) illustration of the notion of ‘rate of change’ and can provoke discussion on other key concepts such as increasing and decreasing function, and the first and second derivatives (Berry *et al.*, 1994).

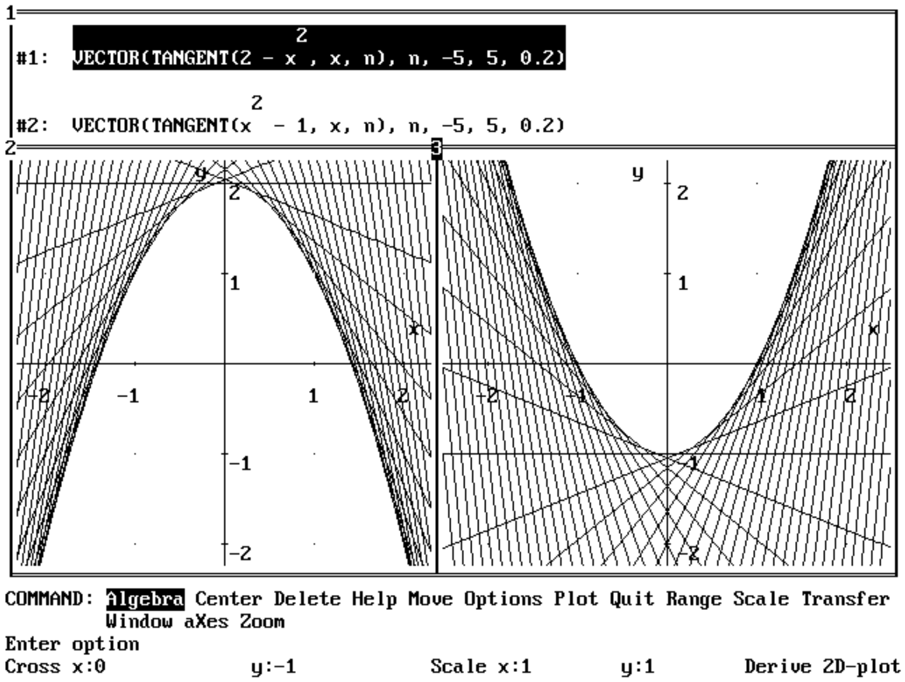

Figure 2. Illustrating the notion of ‘rate-of-change’ to beginning calculus students

Students often have difficulties visualizing graphs of two variables. *DERIVE* can display these functions as surfaces in three dimensional space using the ‘3D-Plot’ facility (figure 3). Different views of the same surface can be generated, providing the student with a visual appreciation of these functions. The graphical representations can be reinforced before the more formal symbolism of partial differentiation is introduced. Again, as with the precalculus exercises, students are encouraged to confirm conjectures discovered visually with their algebraic counterparts.

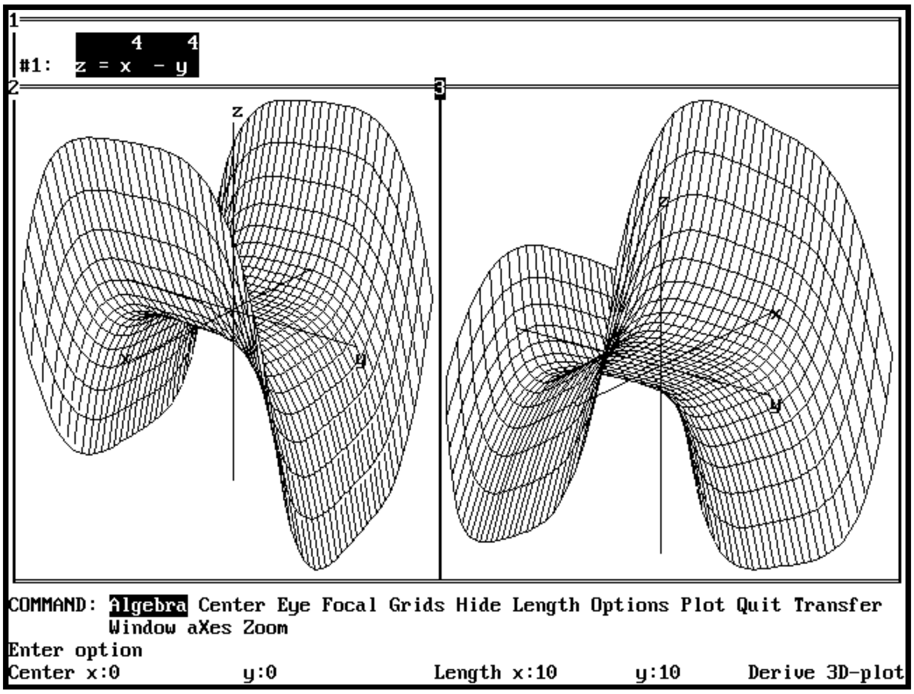

Figure 3. Visualizing functions in 3-dimensional space

3. Using *DERIVE* as an 'Applications' Tool

DERIVE was used as a tutorial aid by computer science students for one hour per week in a computer laboratory. Each student had access to a computer. The emphasis of the tutorials was on 'applications' and 'real world' problem solving activities which were introduced to consolidate the material covered in the four hours of lecture classes each week. As an example, consider a problem for maximising the volume '*V*' of a cylinder with a given surface area. The formula may look like $V = \frac{r(471 - 2\pi r^2)}{2}$. Arriving at this formula requires skill in translating sentences into mathematical expressions, and conceptual understanding which is crucial to the successful solution of the problem. Even if these two obstacles are overcome, the solution to the equation can prove time-consuming and leads to frustration. *DERIVE* can facilitate the solution process through its computational ability and its graph sketching facility (figure 4). The first derivative may be determined and plotted (bottom left of screen in figure 4) to investigate critical points. The notion of 'domain' and 'range' must be understood so that an appropriate scale can be determined before the graph is displayed.

Differential equations model a wide variety of phenomena. Consider the differential equation

$\frac{dv}{dt} = kv$ which relates the rate of change of real estate prices with its value at any one time.

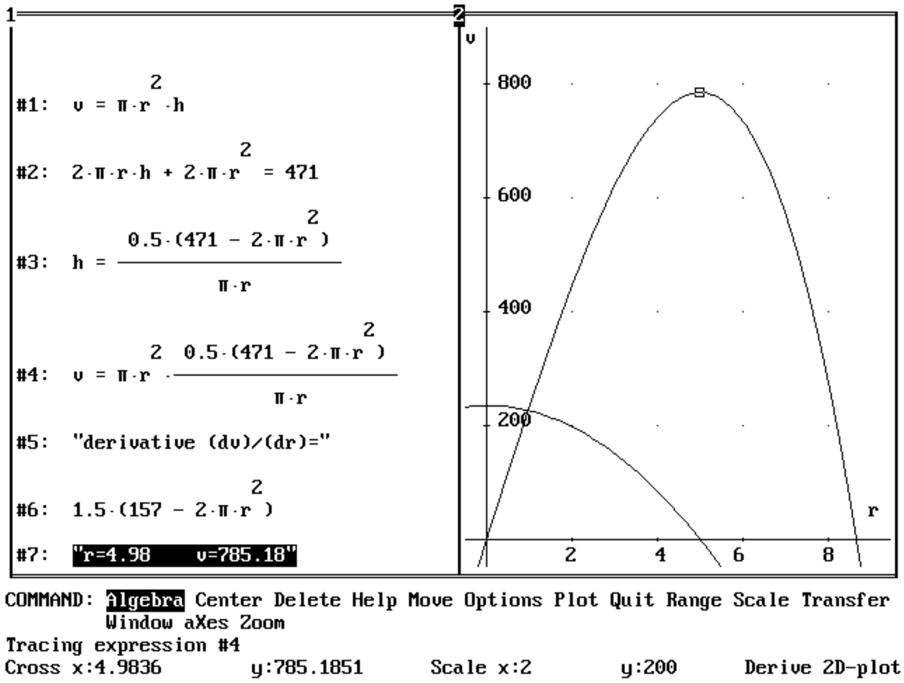

Figure 4. Maximising the volume of a cylinder: concentrating on the concepts

Whilst the algorithm for its solution is reasonably straightforward, the computations involved can be particularly time consuming. Even if students can deal with the ensuing analysis, graphing the function can become an almost insurmountable task. *DERIVE* provides assistance by carrying out the computations and sketching the graph. This leaves the student with more time to interpret the solution and focus on conceptual understanding.

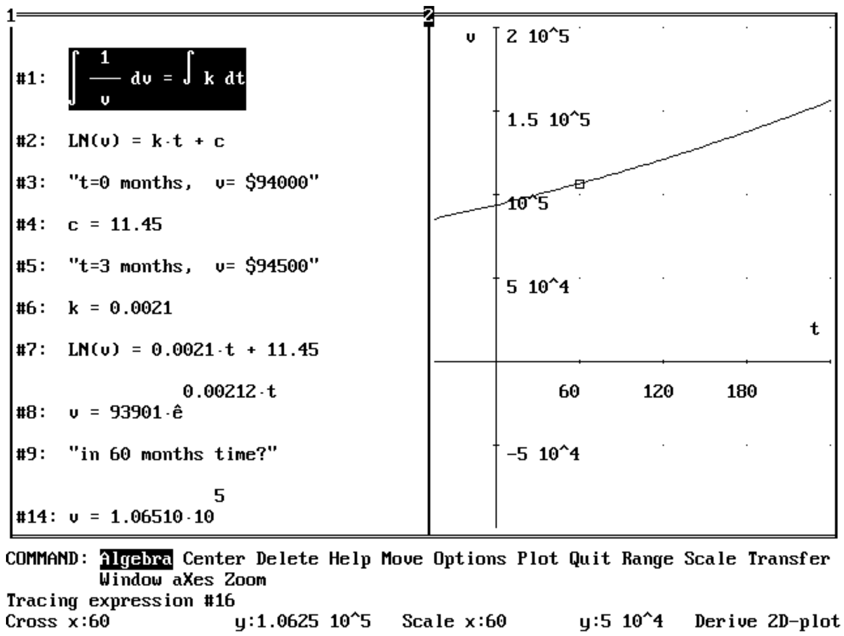

Figure 5. Modelling a problem relating to real estate using differential equations

4. Using *DERIVE* as a Self-paced Exploratory Tool

A group of engineering students repeating a mathematics subject spent half of their allotted weekly subject working in a computer laboratory. The emphasis of the *DERIVE* component was on self-paced, exploratory activities, with maximum student involvement and minimum instructor intervention. As an introductory activity, students explored exponential and logarithmic functions in a way that gave them an intuitive feel for the functions they were generating. The general behaviour of these functions is shown in figure 6 (Berry *et al.,* 1993). Growth and decay graphs can be plotted and the inverse nature of exponential and logarithmic functions clearly observed. Students are encouraged to explore, through *DERIVE's* graphing window, numerous functions of their own.

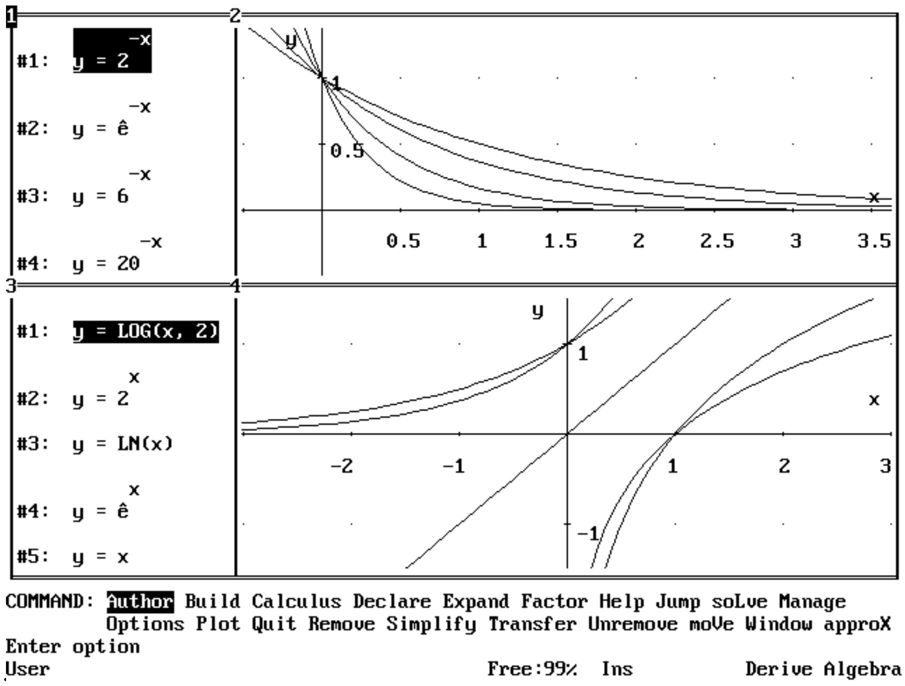

Figure 6 Exploring exponential and logarithmic functions

Translations of 'stock' graphs can be investigated through dilations, horizontal and vertical shifts, with minimum and maximum values located in the graphing window (figure 7). Students are encouraged to experiment, and predict general behaviours for the functions that they are investigating. The exercise, whilst of an elementary nature, gives a number of these reluctant learners confidence, and encourages them to pursue more advanced functions.

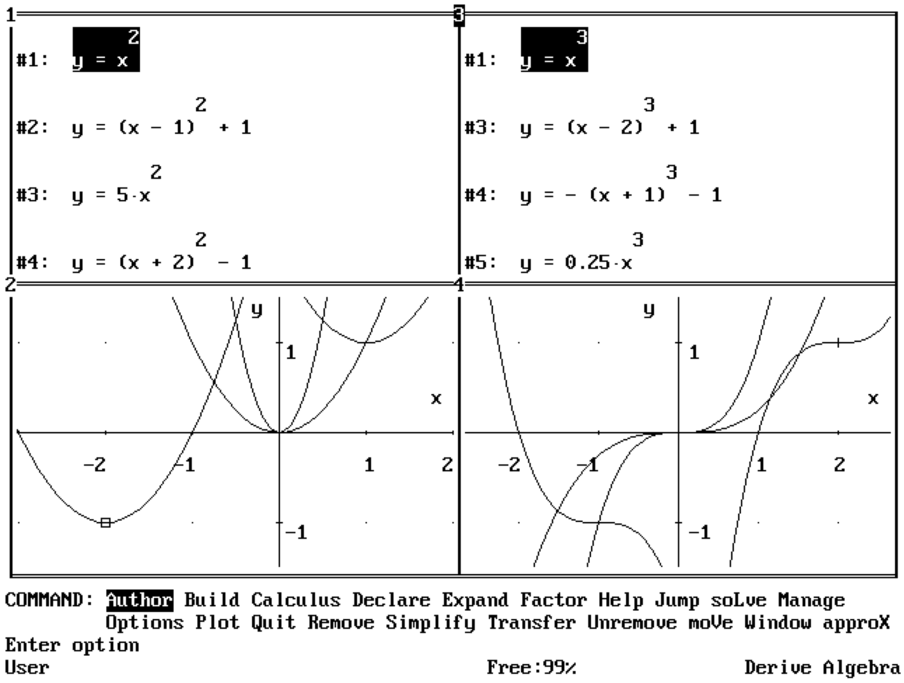

Figure 7. Investigating function transformation problems

5. Discussion and Conclusions

5.1 Learning Styles

Most topics covered by the students favour both an algebraic as well as a visual representation. This allows students to concentrate on a representation that is congruent with their particular learning style. Students have been found to display a wide variety of learning styles, not least of which are those that prefer to process information visually (Lean and Clements, 1981). If students are being exposed to a symbolically-rich curriculum then the visualizers are facing an additional cognitive load. *DERIVE* allows those with a predisposition towards visual learning to make sense of an otherwise ‘difficult’ concept by concentrating on the graphing window.

5.2 Learning to visualize

Whilst using visualization to enhance learning in mathematics is long overdue, there has been little research into just how students learn to apply their visual skills. Learning to visualize is not easy. Why has the analytic approach to learning mathematics always been preferred to its graphical counterpart? Instructors must learn how to communicate mathematics visually. Students need to appreciate the richness of ideas that a diagram can offer, and not revert to symbolic reasoning, even when the algebraic mode is harder (Eisenberg and Dreyfus, 1994). In both the demonstration and self-paced classes, many concepts were taught primarily through the graphing window, with the necessary symbolic links made at a later stage. This is not to say that symbolic reasoning should be relegated to a support role! As long as the necessary visual and symbolic attributes of a concept are emphasized and integrated, then it is the belief of the author that mathematical understanding will be enhanced.

5.3 Conceptual versus procedural learning

CASs, when used thoughtfully, allow students to concentrate on conceptual development. Studies have shown (Heid, 1988; Palmiter, 1991) that computer-based courses which emphasize concepts over procedures can lead to improved performance. In both the ‘applications’ and exploratory classes there were many cases of weaker students making rapid progress despite their poor algorithmic skills. The sequential nature of most topics has had serious repercussions on those who struggled with their mathematics at an early age. Resequencing the skills so that the computer can make the necessary calculations provides these learners with a renewed confidence in their ability to solve problems at this level. This sense of achievement can also motivate the 'reluctant learner' to examine his / her algebraic skill deficiencies. It may be that years of frustration over reasoning in a symbolic mode has so undermined their mathematical confidence that the opportunity to access this knowledge through a new medium is a most attractive proposition. These newly acquired skills have now become tools to understand concepts, not tedious manipulations in their own right.

5.4 Exploration, enthusiasm and interaction

While all three *DERIVE* environments had much to offer in enhancing students' understanding of mathematics, the 'applications' and exploratory classes clearly indicated that students demonstrated a particular involvement and enthusiasm for the problems undertaken. The atmosphere was less formal, cooperative learning was encouraged, and the instructor was able to observe first-hand the obstacles that were confronting students. The 'real world' nature of the applications classes provided the necessary motivation for learning. At last the 'Why are we doing this?' inquiry can be answered with confidence!

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